

# TWO-PARTICLE ANDERSON LOCALIZATION AT LOW ENERGIES

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**ABSTRACT.** We prove exponential spectral localization in a two-particle lattice Anderson model, with a short-range interaction and external random i.i.d. potential, at sufficiently low energies. The proof is based on the multi-particle multi-scale analysis developed earlier by Chulaevsky and Suhov [4] in the case of high disorder. Our method applies to a larger class of random potentials than in Aizenman and Warzel [2] where dynamical localization was proved with the help of the fractional moment method.

## 1. INTRODUCTION. MAIN RESULT

Consider the lattice  $\mathbb{Z}^d \times \mathbb{Z}^d \cong \mathbb{Z}^{2d}$ ,  $d \geq 1$ . We denote  $\mathbb{D} = \{\mathbf{x} \in \mathbb{Z}^{2d} : \mathbf{x} = (x, x)\}$  and  $[[a, b]] := [a, b] \cap \mathbb{Z}$ . Vectors  $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$  will be identified with configurations of two distinguishable quantum particles in  $\mathbb{Z}^d$ . We denote by  $|\cdot|$  the max-norm  $\|\cdot\|_\infty$ , namely for  $x = (x^1, \dots, x^{2d}) \in \mathbb{Z}^{2d}$ ,

$$|x| = \max_{1 \leq i \leq 2d} |x^i|$$

and

$$|x|_1 = \sum_{i=1}^{2d} |x^i|.$$

**The two-particle model.** We study a system of two interacting lattice quantum particles in a disordered environment, described by a random Hamiltonian  $\mathbf{H}_{V,\mathbf{U}}(\omega)$ , acting in the Hilbert space  $\ell^2(\mathbb{Z}^{2d})$ , of the form

$$\mathbf{H}_{V,\mathbf{U}}(\omega) = -\Delta + \sum_{j=1,2} V(x_j, \omega) + \mathbf{U}, \quad (1.1)$$

where  $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$ ,  $\Delta$  is the nearest-neighbor laplacian on  $\mathbb{Z}^{2d}$ ,

$$\Delta \Psi(\mathbf{x}) = \sum_{\substack{\mathbf{y} \in \mathbb{Z}^{2d} \\ |\mathbf{y}|_1=1}} (\Psi(\mathbf{x} + \mathbf{y}) - \Psi(\mathbf{x})), \quad (1.2)$$

$V: \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}$  is a random field with i.i.d. (independent and identically distributed) values on  $\mathbb{Z}^d$ , relative to some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $\mathbf{U}$  is the multiplication operator by a function  $\mathbf{U}(\mathbf{x}) = \mathbf{U}(x_1, x_2)$  which we assume bounded (but not necessarily symmetric).

Aizenmann and Warzel [2] proved by the fractional moment method — introduced in [1] for single-particle systems — the spectral and dynamical localization at low energies

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for such Hamiltonians under the assumption that the marginal probability distribution of the i.i.d. random field  $V$  admits a bounded probability density  $\rho_V$ , satisfying some additional conditions.

In this paper, using Multi-Scale Analysis (MSA) as in [4], we prove exponential localization at low energies under the much weaker assumption of log-Hölder continuity of the marginal distribution function  $F_V$  of the field  $V$ .

*Assumption on  $V$ .* Specifically, we require that for some  $\beta \in (0, 1)$ , some large enough  $q_0 > 0$  and all sufficiently large  $L > 0$ ,

$$\sup_{a \in \mathbb{R}} \mathbb{P} \left\{ V(0, \omega) \in [a, a + e^{-L^\beta}] \right\} \leq L^{-q_0}. \quad (1.3)$$

*Assumptions on  $\mathbf{U}$ .* The interaction potential  $\mathbf{U}$  is assumed to be bounded, non-negative and to satisfy the following short-range condition:

There exists  $0 \leq r_0 < +\infty$  such that

$$|x_1 - x_2| > r_0 \implies \mathbf{U}(x_1, x_2) = 0. \quad (1.4)$$

*Remark.* The assumption of non-negativity of the interaction potential is not essential for our main result (Theorem 1) on Anderson localization for two-particle systems. However, it allows to simplify the adaptation of the two-particle MSA scheme proposed in [4] to the case of weak disorder at low energies. We address a more general class of interacting  $N$ -particle Anderson models at low energies, with any  $N \geq 2$ , in a separate paper [7].

We denote by  $\sigma(\mathbf{H}(\omega))$  the spectrum of  $\mathbf{H}(\omega)$ . It follows from our assumptions and from well-known results that the quantity

$$E^0 := \inf \sigma(\mathbf{H}(\omega))$$

is non-random, although it may be infinite, e.g., for gaussian random potentials.

Given an arbitrary finite lattice cube

$$\mathbf{C}_L(\mathbf{u}) := \{\mathbf{x} \in \mathbb{Z}^{2d} : |\mathbf{x} - \mathbf{u}| \leq L\}$$

we define its exterior and interior boundaries, respectively by

$$\partial^+ \mathbf{C}_L(\mathbf{u}) = \{\mathbf{v} \in \mathbb{Z}^{2d} \mid \text{dist}(\mathbf{v}, \mathbf{C}_L(\mathbf{u})) = 1\}, \quad (1.5)$$

$$\partial^- \mathbf{C}_L(\mathbf{u}) = \{\mathbf{v} \in \mathbb{Z}^{2d} \mid \text{dist}(\mathbf{v}, \mathbb{Z}^{2d} \setminus \mathbf{C}_L(\mathbf{u})) = 1\}. \quad (1.6)$$

We will consider the finite-volume approximation  $\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}$  of  $\mathbf{H}$  defined by

$$\mathbf{H}_{\mathbf{C}_L(\mathbf{u})} := \mathbf{H}_{\mathbf{C}_L(\mathbf{u})}^D \quad (1.7)$$

$$:= \mathbf{H}|_{\ell^2(\mathbf{C}_L(\mathbf{u}))} \text{ with Dirichlet boundary conditions on } \partial^+ \mathbf{C}_L(\mathbf{u}).$$

Our main result is

**Theorem 1** (localization at low energies). *Let  $\mathbf{H}_{V, \mathbf{U}}(\omega)$  be the random hamiltonian defined in (1.1). Suppose that  $V$  is an i.i.d. random field satisfying (1.3), and that the potential interaction  $\mathbf{U}$  is bounded, non-negative and satisfies (1.4). Let  $E^0 = \inf \sigma(\mathbf{H})$ .*

*Then there exists  $E^* > E^0$  such that*

- (i) *the spectrum of  $\mathbf{H}(\omega)$  in  $[E^0, E^*]$  is pure point,*
- (ii) *all its eigenfunctions  $\Psi_n(\omega)$  with eigenvalues  $E_n(\omega) \in [E^0, E^*]$  are exponentially decaying at infinity with a positive non-random rate of decay  $m > 0$ :*

$$|\Psi_n(\mathbf{x})| \leq C_n(\omega) e^{-m|\mathbf{x}|}. \quad (1.8)$$

To prove Theorem 1, we use an adaptation of the MSA to the two-particle interacting systems, following [4]. Given a finite cube  $\mathbf{C}_L(\mathbf{u}) \subset \mathbb{Z}^{2d}$ , introduce the resolvent of the operator  $\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}$ ,

$$\mathbf{G}_{\mathbf{C}_L(\mathbf{u})}(E) := (\mathbf{H}_{\mathbf{C}_L(\mathbf{u})} - E)^{-1}, \quad E \in \mathbb{R} \setminus \sigma(\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}). \quad (1.9)$$

Its matrix elements  $\mathbf{G}_{\mathbf{C}_L(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)$  in the canonical basis  $\{\delta_{\mathbf{x}}\}$  in  $\ell^2(\mathbb{Z}^{2d})$  are usually called the Green functions of the operator  $\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}$ :

$$\mathbf{G}_{\mathbf{C}_L(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E) = \left\langle (\mathbf{H}_{\mathbf{C}_L(\mathbf{u})} - E)^{-1} \delta_{\mathbf{x}}, \delta_{\mathbf{y}} \right\rangle, \quad \mathbf{x}, \mathbf{y} \in \mathbf{C}_L(\mathbf{u}). \quad (1.10)$$

According to the general MSA approach, the exponential localization will be derived from Theorem 3 below. To formulate it, we introduce the following notion.

**Definition 1** ( $(E, m)$ -singular). Let  $m > 0$  and  $E \in \mathbb{R}$ . A cube  $\mathbf{C}_L(\mathbf{u}) \subset \mathbb{Z}^{2d}$  is called  $(E, m)$ -non-singular ( $(E, m)$ -NS) if

$$\max_{\mathbf{v} \in \partial^+ \mathbf{C}_L(\mathbf{u})} |\mathbf{G}_{\mathbf{C}_L(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E)| \leq e^{-mL}. \quad (1.11)$$

Otherwise, it is called  $(E, m)$ -singular ( $(E, m)$ -S).

Let  $\mathbf{S}$  be the symmetry  $\mathbf{x} = (x_1, x_2) \mapsto \mathbf{S}\mathbf{x} = (x_2, x_1)$  in the lattice  $\mathbb{Z}^{2d} = \mathbb{Z}^d \times \mathbb{Z}^d$  (with  $x_1, x_2 \in \mathbb{Z}^d$ ). The “symmetrized distance” is defined in  $\mathbb{Z}^{2d}$  by

$$d_{\mathbf{S}}(\mathbf{x}, \mathbf{y}) = \min \{ |\mathbf{x} - \mathbf{y}|, |\mathbf{S}(\mathbf{x}) - \mathbf{y}| \}. \quad (1.12)$$

**Definition 2** ( $\ell$ -distant). Two subsets  $\mathbf{A}, \mathbf{B} \subset \mathbb{Z}^{2d}$  are called  $\ell$ -distant if

$$d_{\mathbf{S}}(\mathbf{A}, \mathbf{B}) > 8\ell.$$

The multiscale analysis is based on a length scale  $\{L_k\}_{k \geq 0}$  which is chosen as follows.

**Definition 3** (length scale). The length-scale  $\{L_k\}_{k \geq 0}$  is a sequence of integers defined by the initial length-scale  $L_0 > 2$ , and by the recurrence relation  $L_{k+1} = \lfloor L_k^\alpha \rfloor$ ,  $k \geq 0$  where  $1 < \alpha < 2$  is some fixed number. In this paper,  $\alpha = 3/2$ .

The length scale  $\{L_k\}_{k \geq 0}$  is assumed to be chosen at the beginning of the multiscale analysis, except that in the course of the analysis it is often required that  $L_0$  be large enough.

**Definition 4.** Given a positive number  $m_0 > 0$ , we define a positive sequence  $m_k$  depending upon a positive number  $\gamma > 0$  by

$$m_k = m_0 \prod_{j=1}^k (1 - \gamma L_j^{-1/2}), \quad k \geq 1. \quad (1.13)$$

It will be assumed that  $L_0$  is large enough so that

$$\prod_{j=1}^{\infty} (1 - \gamma L_j^{-1/2}) \geq \frac{1}{2}.$$

We introduce the following property of pairs of two-particle cubes of size  $L_k$ :

**(DS.k).** For any pair of  $L_k$ -distant cubes  $\mathbf{C}_{L_k}(\mathbf{u})$  and  $\mathbf{C}_{L_k}(\mathbf{v})$ :

$$\mathbb{P} \{ \exists E \in I : \mathbf{C}_{L_k}(\mathbf{u}) \text{ and } \mathbf{C}_{L_k}(\mathbf{v}) \text{ are } (E, m_k)\text{-S} \} \leq L_k^{-2p}, \quad (1.14)$$

where  $p > 12d$ , and  $I = [E^0, E^*]$  with  $E^* > E^0$ , are fixed.

*Comment.* This property depends on  $m_k, p, L_0$ , and  $E^*$ . Therefore, it would be better to use a more precise notation, like  $(\mathbf{DS}.k)_{m_k, p, L_0, E^*}$  or even simply  $(\mathbf{DS}.k)_{m_k}$ .

The analogous property for one-particle cubes in  $\mathbb{Z}^d$  is as follows:

(DS.k). For any pair of *disjoint* cubes  $C_{L_k}(u)$  and  $C_{L_k}(v)$ :

$$\mathbb{P} \{ \exists E \in I : C_{L_k}(u) \text{ and } C_{L_k}(v) \text{ are } (E, m_k)\text{-S} \} \leq L_k^{-2\tilde{p}}, \quad (1.15)$$

where  $\tilde{p} > 2d$ , and  $I = [E^0, E^*]$  with  $E^* > E^0$ , are fixed.

For a single-particle random Hamiltonian of the form  $H = -\Delta + V(x, \omega)$  in  $\ell^2(\mathbb{Z}^d)$  we have the well known result:

**Theorem 2** (one-particle estimate). *Let  $\tilde{p} > 2d$  be fixed. Then, provided  $L_0$  is large enough, there exists  $E^* = E^*(\tilde{p}) > E^0$  such that  $(\mathbf{DS}.k)_{m_k, p, L_0, E^*}$  holds true for all  $k \geq 0$ .*

*Proof.* See, e.g., [4, Result 9.8 & Chapters 10-11].  $\square$

Here we will prove the same result for two-particle random Hamiltonian and we will be allowed to use Theorem 2 in which we assume that the exponent  $\tilde{p}$  satisfies:

$$\tilde{p} > \frac{9}{4}p + \frac{15}{2}d.$$

**Theorem 3** (two-particle estimate). *Let  $p > 12d$  be fixed. Then, provided  $L_0$  is large enough, there exist  $E^* = E^*(p) > E^0$  such that  $(\mathbf{DS}.k)_{m_k, p, L_0, E^*}$  holds true for all  $k \geq 0$ .*

The proof is based on induction in  $k$ . Note that the initial length-scale estimate (for  $L_0$  sufficiently large) uses the Combes–Thomas estimate and the Lifshitz tails phenomenon, essentially in the same way as for single-particle models [9, 10]. In fact, the single- or multi-particle structure of the potential energy is not crucial for such a bound. The inductive step is performed almost in the same way as in the case of high disorder (see [4]). It uses Wegner-type estimates proved in [3] (see [11] for the original Wegner estimate). Note, however, that unlike the high disorder regime, the value of the mass  $m > 0$  may be small, depending upon the amplitude of the random potential  $V$ . Namely, if the random external potential has the form  $gV(x; \omega)$ , then the value of the mass  $m = m(g) \rightarrow 0$  as  $|g| \rightarrow 0$ .

The derivation of the spectral localization from the bounds of the multi-particle MSA can be obtained in the same way as in the case of high disorder. The following statement is a reformulation of [4, Theorem 1.2]. In turn, the main idea of the proof goes back to [5]. In a different form, a similar argument appears already in [8].

**Theorem 4.** *Suppose that  $(\mathbf{DS}.k)$  holds true for some  $E^* > E^0$ . Then, for  $\mathbb{P}$ -almost all  $\omega$ ,*

- (i) *the spectrum of  $\mathbf{H}(\omega)$  in  $(-\infty, E^*]$  is pure point,*
- (ii) *there exists a non-random number  $m > 0$  such that all eigenfunctions  $\Psi_n(\omega)$  of  $\mathbf{H}(\omega)$  with eigenvalues  $E_n(\omega) \leq E^*$  are exponentially decaying at infinity with rate  $m$ :*

$$|\Psi_n(\mathbf{x})| \leq C_n(\omega) e^{-m|\mathbf{x}|}. \quad (1.16)$$

*Proof.* See [4, Theorem 1.2].  $\square$

Theorem 1 derives clearly from Theorem 4 and Theorem 3. Therefore, it only remains to prove Theorem 3, i.e., to check property **(DS.k)** for all  $k \geq 0$ .

The results of this paper were announced in [6].

## 2. THE TWO-PARTICLE MSA SCHEME

We now outline the two-particle MSA which is used for the proof of Theorem 3.

The following definition depends on a parameter  $0 < \beta < 1$ . For our purposes, we take  $\beta = 1/2$ , but we keep  $\beta$  in all formulae to show the dependence on this parameter.

**Definition 5** (*E-resonant*). Let  $E \in \mathbb{R}$  be given. A cube  $\mathbf{C}_L(\mathbf{v}) \subset \mathbb{Z}^{2d}$  of size  $L \geq 2$  is called *E-resonant* (*E-R*) if

$$\text{dist} [E, \sigma (\mathbf{H}_{\mathbf{C}_L(\mathbf{v})})] < e^{-L^\beta}. \quad (2.1)$$

Otherwise it is called *E-non-resonant* (*E-NR*).

The next definition depends on the parameter  $\alpha > 1$  which governs the length scale of our multiscale analysis. For our purposes, we take  $\alpha = 3/2$ , but we keep  $\alpha$  in all formulae.

**Definition 6** (*E-completely non-resonant*). Let  $E \in \mathbb{R}$  be given. A cube  $\mathbf{C}_L(\mathbf{v}) \subset \mathbb{Z}^{2d}$  of size  $L \geq 2$  is called *E-completely non-resonant* (*E-CNR*) if it does not contain any *E-R* cube of size  $\geq L^{1/\alpha}$ . In particular,  $\mathbf{C}_L(\mathbf{v})$  is itself *E-NR*.

Given  $L_0 > 2$ , we introduce the following properties **(W1)** and **(W2)** of the random Hamiltonians  $\mathbf{H}_{\mathbf{C}_\ell}$ ,  $\ell \geq L_0$ :

**(W1)**. For any cube  $\mathbf{C}_\ell(\mathbf{x})$  of size  $\ell \geq L_0$  and any  $E \in \mathbb{R}$ ,

$$\mathbb{P} \{ \mathbf{C}_\ell(\mathbf{x}) \text{ is not } E\text{-CNR} \} < \ell^{-q}, \quad (2.2)$$

where  $q > 4p$  and  $L_0 > 2$  are given.

**(W2)**. For any  $\ell$ -distant cubes  $\mathbf{C}_\ell(\mathbf{x})$  and  $\mathbf{C}_\ell(\mathbf{y})$  of size  $\ell \geq L_0$ ,

$$\mathbb{P} \{ \exists E \in \mathbb{R} : \text{neither } \mathbf{C}_\ell(\mathbf{x}) \text{ nor } \mathbf{C}_\ell(\mathbf{y}) \text{ is } E\text{-CNR} \} < \ell^{-q} \quad (2.3)$$

where  $q > 4p$  and  $L_0 > 2$  are given.

*Comment.* These properties depend on  $q$  and  $L_0$ . Hence, better notations would be  $(\mathbf{W1})_{q, L_0}$  and  $(\mathbf{W2})_{q, L_0}$ .

**Lemma 1** (Wegner-type estimates). *Let  $q_1, q_2 > 0$  and  $L_0 > 0$  be given. Under assumptions (1.3) on the random potential  $V(x, \omega)$  and assumption (1.4) on the interaction potential  $\mathbf{U}$ , properties **(W1)** for  $q = q_1$  and **(W2)** for  $q = q_2$  hold true for any  $\ell \geq L_0$  provided  $L_0$  and  $q_0$  are large enough.*

*Proof.* (i) We first prove **(W1)** for a given  $q = q_1$  provided  $L_0$  and  $q_0$  are large enough. Let  $\ell \geq L_0$ ,  $\mathbf{C}_\ell(\mathbf{x})$  and  $E \in \mathbb{R}$  be fixed. We have:

$$\begin{aligned} & \mathbb{P} \{ \mathbf{C}_\ell(\mathbf{x}) \text{ is not } E\text{-CNR} \} \\ &= \mathbb{P} \left\{ \exists \mathbf{y} \in \mathbf{C}_\ell(\mathbf{x}), \exists \ell' : \ell^{1/\alpha} \leq \ell' \leq \ell, \mathbf{C}_{\ell'}(\mathbf{y}) \text{ is } E\text{-R} \right\} \\ &\leq |\mathbf{C}_\ell(\mathbf{x})| (\ell + 1) \times \mathbb{P} \{ \mathbf{C}_{\ell'}(\mathbf{y}) \text{ is } E\text{-R} \} \\ &\leq (2\ell + 1)^{2d} (\ell + 1) \times \mathbb{P} \left\{ \text{dist} [E, \sigma (\mathbf{H}_{\mathbf{C}_{\ell'}(\mathbf{y})})] < e^{-\ell'^\beta} \right\} \end{aligned}$$

then by the “basic” two-particle one-volume Wegner estimate [3, Theorem 1]

$$\leq (2\ell + 1)^{2d+1} \times |\mathbf{C}_{\ell'}(\mathbf{y})| \times |C_{\ell'}(y_1)| \times \sup_{a \in \mathbb{R}} \mathbb{P} \left\{ V(0, \omega) \in \left[ a, a + 2e^{-\ell'^\beta} \right] \right\}$$

which gives, using assumption (1.3), and provided  $L_0$  and  $q_0$  are large enough

$$\leq (2\ell + 1)^{5d+1} \ell^{-q_0/\alpha} < \ell^{-q_1}.$$

(ii) Now we prove **(W2)** for a given  $q = q_2$  provided  $L_0$  and  $q_0$  are large enough. Let  $\ell \geq L_0$  and  $\mathbf{C}_\ell(\mathbf{x})$ ,  $\mathbf{C}_\ell(\mathbf{y})$  be fixed. We have:

$$\begin{aligned} & \mathbb{P} \{ \exists E \in \mathbb{R} : \text{neither } \mathbf{C}_\ell(\mathbf{x}) \text{ nor } \mathbf{C}_\ell(\mathbf{y}) \text{ is } E\text{-CNR} \} \\ &= \mathbb{P} \{ \exists E \in \mathbb{R}, \exists \mathbf{u} \in \mathbf{C}_\ell(\mathbf{x}), \exists \mathbf{v} \in \mathbf{C}_\ell(\mathbf{y}), \exists \ell_1, \ell_2 \text{ with } \ell^{1/\alpha} \leq \ell_1, \ell_2 \leq \ell : \\ & \quad \mathbf{C}_{\ell_1}(\mathbf{u}) \text{ and } \mathbf{C}_{\ell_2}(\mathbf{v}) \text{ are } E\text{-R} \} \\ &\leq |\mathbf{C}_\ell(\mathbf{x})| |\mathbf{C}_\ell(\mathbf{y})| (\ell + 1)^2 \times \mathbb{P} \{ \exists E \in \mathbb{R} : \mathbf{C}_{\ell_1}(\mathbf{u}), \mathbf{C}_{\ell_2}(\mathbf{v}) \text{ are } E\text{-R} \} \\ &\leq (2\ell + 1)^{4d} (\ell + 1)^2 \times \mathbb{P} \{ \exists E \in \mathbb{R} : \text{dist}[E, \sigma(\mathbf{H}_{\mathbf{C}_{\ell_j}(\mathbf{u})})] < e^{-\ell_j^\beta}, j = 1, 2 \} \\ &\leq (2\ell + 1)^{4d} (\ell + 1)^2 \times \mathbb{P} \{ \exists E \in \mathbb{R} : \text{dist}[E, \sigma(\mathbf{H}_{\mathbf{C}_{\ell_j}(\mathbf{u})})] < e^{-(\ell_1 \wedge \ell_2)^\beta}, j = 1, 2 \} \\ &\leq (2\ell + 1)^{4d} (\ell + 1)^2 \times \mathbb{P} \{ \text{dist}[\sigma(\mathbf{H}_{\mathbf{C}_{\ell_1}(\mathbf{u})}), \sigma(\mathbf{H}_{\mathbf{C}_{\ell_2}(\mathbf{v})})] < 2e^{-(\ell_1 \wedge \ell_2)^\beta} \} \end{aligned}$$

then by the basic two-particle Wegner-type estimate [3, Theorem 2]

$$\begin{aligned} &\leq (2\ell + 1)^{4d} (\ell + 1)^2 \times |\mathbf{C}_{\ell_1}(\mathbf{u})| \times |\mathbf{C}_{\ell_2}(\mathbf{v})| \times \max \{ |C_{\ell_1}(u_1)|, |C_{\ell_2}(v_1)| \} \\ &\quad \times \sup_{a \in \mathbb{R}} \mathbb{P} \left\{ V(0, \omega) \in \left[ a, a + 4e^{-(\ell_1 \wedge \ell_2)^\beta} \right] \right\} \end{aligned}$$

which gives, using assumption (1.3), and provided  $L_0$  and  $q_0$  are large enough

$$\leq (2\ell + 1)^{9d+2} \ell^{-q_0/\alpha} < \ell^{-q_2}. \quad \square$$

We consider now a property which serves as replacement of **(DS.0)**.

**(S.0).** For any cube  $\mathbf{C}_{L_0}(\mathbf{x}) \subset \mathbb{Z}^{2d}$ ,

$$\mathbb{P} \{ \exists E \in [E^0, E^*] : \mathbf{C}_{L_0}(\mathbf{x}) \text{ is } (E, m_0)\text{-S} \} < L_0^{-2p}, \quad (2.4)$$

where  $E^* > E^0$ ,  $m_0 > 0$  and  $L_0 \geq 2$  are given.

Obviously, property **(S.0)** implies property **(DS.0)**, so we focus on the former. Property **(S.0)** is proven in [4] in the case of high disorder. Our proof presented here is completely different. It uses the Combes–Thomas estimate and the well-known “Lifshitz tails” phenomenon.

We start with the Combes–Thomas estimate, formulated for fairly general discrete Schrödinger operators  $H = -\Delta + W$ , acting in a finite-dimensional Hilbert space  $\ell^2(\Lambda)$ ,  $\Lambda \subset \mathbb{Z}^n$  finite, where  $\Delta$  is the nearest-neighbor discrete Laplacian. It is deterministic, and the structure of the potential  $W(x)$  is irrelevant. This allows to apply it to the two-particle Hamiltonian  $\mathbf{H} = -\Delta + \mathbf{W}$ , acting in the space  $\ell^2(\mathbf{C}_L(\mathbf{u}))$ , with  $\mathbf{C}_L(\mathbf{u}) \subset \mathbb{Z}^{2d}$  and  $\mathbf{W}(\mathbf{x}) = V(x_1; \omega) + V(x_2; \omega) + \mathbf{U}(\mathbf{x})$ .

**Lemma 2** (Combes–Thomas estimate). *Let  $\mathbf{H}: \ell^2(\Lambda) \rightarrow \ell^2(\Lambda)$ ,  $\Lambda \subset \mathbb{Z}^n$ . Suppose that  $E \in \mathbb{R}$  satisfies  $\text{dist}(E, \sigma(\mathbf{H})) = \delta \leq 1$ . Then, for any  $x, y \in \Lambda$ ,*

$$|(\mathbf{H} - E)^{-1}(x, y)| \leq \frac{2}{\delta} e^{-\frac{\delta}{12n}|x-y|_1} \leq \frac{2}{\delta} e^{-\frac{\delta}{12n}|x-y|}. \quad (2.5)$$

*Proof.* See [9, Theorem 11.2].  $\square$

Next, we need the following statement, also applying to a general Schrödinger operator on finite subsets of a lattice of arbitrary dimension  $n$ . It summarizes well-known results (cf. [9, 10] and references therein) from the spectral theory of a random one-particle Schrödinger operator  $H(\omega) = -\Delta + V(x; \omega)$ .

**Lemma 3.** *Let  $H(\omega) = -\Delta + V(x; \omega)$  be a random Schrödinger operator, with non-negative i.i.d. random potential  $V(\cdot; \omega)$ , restricted to a cube  $C_\ell(u) \subset \mathbb{Z}^d$  with Dirichlet boundary conditions. Suppose the random variables  $V(x; \omega)$  are non-constant and non-negative. Then for any  $C > 0$  and arbitrary large  $L_0 > 0$ , there exists  $c > 0$  such that the lowest eigenvalue  $E_0(\omega)$  of  $H_{C_{L_0}(u)}(\omega)$  satisfies the bound*

$$\mathbb{P} \left\{ E_0(\omega) \leq 2CL_0^{-1/2} \right\} \leq e^{-c|C_{L_0}(u)|^{1/4}}. \quad (2.6)$$

*Remark.* This lemma is actually proven for the lowest eigenvalue  $E_0^N(\omega)$  of the finite-volume hamiltonian  $H^N(\omega) = -\Delta + V(x; \omega)$  with Neumann boundary conditions. By Dirichlet–Neumann bracketing, the same bound holds true for the lowest eigenvalue  $E_0(\omega) = E_0^D(\omega)$  of the finite-volume hamiltonian  $H_{C_{L_0}(u)}(\omega) = H_{C_{L_0}(u)}^D(\omega)$  with Dirichlet boundary conditions.

*Proof.* See [9, Estimate (11.16)] which follows from the study of Lifschitz tails, precisely from [9, Estimate (6.10)] and [9, Lemma 6.4].  $\square$

**Lemma 4.** *Consider a random Schrödinger operator  $\mathbf{H}(\omega) = -\Delta + \mathbf{W}(\mathbf{x}; \omega)$  in a lattice cube  $C_\ell(u) \subset \mathbb{Z}^n$  with Dirichlet boundary conditions. Suppose that:*

- (a) *The random external potential  $V(x; \omega)$  and the interaction potential  $\mathbf{U}(x_1, x_2)$  are non-negative.*
- (b) *For any  $\epsilon > 0$ ,  $\mathbb{P} \{V(x; \omega) < \epsilon\} > 0$ , i.e., 0 is the sharp lower bound for the values of the random potential  $V$ .*
- (c) *The random variables  $V(x; \omega)$  are non-constant:  $\mathbb{P} \{V(x; \omega) > 0\} > 0$ .*

*Then for any  $C > 0$  and arbitrary large  $L_0 > 0$  there exist  $c > 0$  such that the lowest eigenvalue  $E_0(\omega)$  of  $\mathbf{H}_{C_{L_0}(u)}(\omega)$  satisfies the bound*

$$\mathbb{P} \left\{ E_0(\omega) \leq 2CL_0^{-1/2} \right\} \leq e^{-c|C_{L_0}(u)|^{1/4}}. \quad (2.7)$$

*Proof.* The interaction potential  $\mathbf{U}$  is non-negative, so that by min-max principle, the lowest eigenvalue  $E_0(\omega)$  of  $\mathbf{H}_{C_{L_0}(u)}(\omega)$  is bounded from below by the lowest eigenvalue  $E_0^N(\omega)$  of operator  $-\Delta + V(x_1; \omega) + V(x_2; \omega)$ . This latter operator can be written as follows:

$$-\Delta + V(x_1; \omega) + V(x_2; \omega) = H^{(1)} \otimes I^{(2)} + I^{(1)} \otimes H^{(2)},$$

where  $H^{(j)} = -\Delta + V(x_j; \omega)$ ,  $j = 1, 2$ . As a result,  $E_0^N(\omega)$  must have the form  $E_0^N(\omega) = E_0^{(1)}(\omega) + E_0^{(2)}(\omega)$ , where  $E_0^{(j)}(\omega)$  is the lowest eigenvalue of  $H^{(j)}$ . Finally,  $E_0^{(1)}(\omega)$  and  $E_0^{(2)}(\omega)$  are non-negative due to the non-negativity of the external potential, so that for any  $s \geq 0$

$$\mathbb{P} \left\{ E_0^N(\omega) \leq s \right\} \leq \mathbb{P} \left\{ E_0^{(1)}(\omega) \leq s \right\}.$$

Now the assertion follows from Lemma 3 applied to the single-particle Schrödinger operator  $H^{(1)}$ .  $\square$

Lemma 4 leads directly to the initial scale estimate for our two-particle model.

**Theorem 5** (initial scale estimate). *For any  $p > 0$ ,  $C > 0$  and  $L_0$  large enough, there exists  $E^* > E^0$  such that properties (S.0) and (DS.0) hold true for some  $m_0 \geq CL_0^{-1/2} > 0$ .*

*Proof.* Use Lemma 4 and Lemma 2.  $\square$

To complete the inductive step of the two-particle MSA, it only remains to prove

**Theorem 6.** *There exists  $0 < L^* < \infty$  such that, for any  $L_0 \geq L^*$  and any  $k \geq 0$ ,*

$$(\mathbf{DS}.k)_{m_k} \implies (\mathbf{DS}.k+1)_{m_{k+1}}.$$

For the proof we introduce

**Definition 7** (diagonal cube). Let  $r_0 > 0$  be as in 1.4 and let

$$\mathbb{D}_{r_0} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^{2d} : |x_1 - x_2| \leq r_0\}. \quad (2.8)$$

A two-particle cube  $\mathbf{C}_L(\mathbf{u})$  is called *diagonal* (D) when  $\mathbf{C}_L(\mathbf{u}) \cap \mathbb{D}_{r_0} \neq \emptyset$ . Otherwise it is called *non-diagonal* (ND).

*Remark.* The interaction potential  $\mathbf{U}$  vanish identically on any non-diagonal cube.

The procedure of deducing property  $(\mathbf{DS}.k+1)$  from  $(\mathbf{DS}.k)$  is done separately for the following three cases:

- (I) Both  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  and  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$  are ND-cubes.
- (II) Both  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  and  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$  are D-cubes.
- (III) One of the cubes is D, while the other is ND.

More precisely:

- (i) In Section 3 we prove  $(\mathbf{DS}.k)^{(I)}$  for any  $k \geq 0$ .
- (ii) In Section 4 we prove  $(\mathbf{DS}.k)^{(I,II)} \implies (\mathbf{DS}.k+1)^{(II)}$ .
- (iii) In Section 5 we prove  $(\mathbf{DS}.k)^{(I,II)} \implies (\mathbf{DS}.k+1)^{(III)}$ .

All cases require using of property (W1) and/or property (W2).

### 3. CASE (I). NON-DIAGONAL PAIR OF SINGULAR CUBES

In this section, we aim to prove  $(\mathbf{DS}.k)$  for any  $k \geq 0$  and any pair of  $L_k$ -distant non-diagonal cubes  $\mathbf{C}_{L_k}(\mathbf{x})$  and  $\mathbf{C}_{L_k}(\mathbf{y})$ . In that case the interaction vanishes and we are mainly reduced to the one-particle case.

Let  $\mathbf{C}_{L_k}(\mathbf{u}) \subset \mathbb{Z}^{2d}$  be a non-diagonal cube,  $\mathbf{u} = (u_1, u_2)$ :

$$\mathbf{C}_{L_k}(\mathbf{u}) = C_{L_k}(u_1) \times C_{L_k}(u_2). \quad (3.1)$$

Since  $\mathbf{U}$  vanishes on the non-diagonal cube  $\mathbf{C}_{L_k}(\mathbf{u})$ , we have, for  $\mathbf{x} = (x_1, x_2) \in \mathbf{C}_{L_k}(\mathbf{u})$ ,

$$(\mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u})}\Psi)(\mathbf{x}) = \sum_{|\mathbf{y}|_1=1} \Psi(\mathbf{x} + \mathbf{y}) + (V(x_1, \omega) + V(x_2, \omega)) \Psi(\mathbf{x}), \quad (3.2)$$

which can be written (take  $\Psi = \Psi_1 \otimes \Psi_2$ )

$$\mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u})} = H_{C_{L_k}(u_1)}^{(1)} \otimes I^{(2)} + I^{(1)} \otimes H_{C_{L_k}(u_2)}^{(2)}. \quad (3.3)$$



Here  $H_{C_{L_k}(u_j)}^{(j)}$  is the single-particle Hamiltonian acting on  $\Psi_j$ ,  $x_j \in C_{L_k}(u_j)$ ,  $j = 1, 2$ :

$$\left(H_{C_{L_k}(u_j)}^{(j)}\Psi_j\right)(x_j) = \sum_{\substack{y_j \in C_{L_k}(u_j) \\ |y_j|=1}} \Psi_j(x_j + y_j) + V(x_j, \omega)\Psi_j(x_j) \quad (3.4)$$

and  $I^{(j)}$  is the identity operator on the complementary space.

In the proof we are using the validity of the bound (DS.k) for one-particle random Schrödinger operators like  $H^{(j)}$  at low energies, provided  $E^*$  is sufficiently close to  $E_0$ :

**Definition 8** (*m-tunnelling*). Let  $I = [E^0, E^*]$  with  $E^* > E^0$  and  $m > 0$  be fixed.

- (i) A *single-particle* cube  $C_{L_k}(u) \subset \mathbb{Z}^d$  is called *m-tunnelling* (*m-T*) if there exists  $E \in I$  and two disjoint cubes  $C_{L_{k-1}}(v_1), C_{L_{k-1}}(v_2) \subset C_{L_k}(u)$  which are  $(E, m)$ -S with respect to an Hamiltonian like  $H^{(j)}$ ,  $j = 1, 2$ . Otherwise it is called *m-non tunnelling* (*m-NT*).
- (ii) A *two-particle* non-diagonal cube  $\mathbf{C}_{L_k}(\mathbf{u}) = C_{L_k}(u_1) \times C_{L_k}(u_2) \subset \mathbb{Z}^{2d}$  is called *m-non-tunnelling* if the single-particle cubes  $C_{L_k}(u_1)$  and  $C_{L_k}(u_2)$  are *m-NT* with respect to  $H^{(1)}$  and  $H^{(2)}$ , respectively. Otherwise, it is called *m-tunnelling*.

The following statement gives a formal description of a property of ND-cubes which will be refer to as (**NDRoNS**) (“Non-Diagonal cubes are Resonant or Non-Singular”).

**Lemma 5** ([4, Lemma 3.2]). Let  $\mathbf{C}_{L_k}(\mathbf{u}) = C_{L_k}(u_1) \times C_{L_k}(u_2) \subset \mathbb{Z}^{2d}$  be a two-particle cube such that

- (i)  $|u_1 - u_2| > 2L_k + r_0$ ,
- (ii)  $\mathbf{C}_{L_k}(\mathbf{u})$  is  $m'$ -NT for some given  $m' > 0$ ,
- (iii)  $\mathbf{C}_{L_k}(\mathbf{u})$  is  $E$ -CNR for some  $E \in \mathbb{R}$ .

Then  $\mathbf{C}_{L_k}(\mathbf{u})$  is  $(E, m)$ -NS, with

$$m = m' - L_k^{-1} \ln(2L_k + 1)^d. \quad (3.5)$$

In particular, if  $L_k^{-1} \ln(2L_k + 1)^d \leq \frac{m'}{2}$  —which is true for sufficiently large  $L_0$ — then  $m \geq \frac{m'}{2}$ .

*Proof.* See [4, Lemma 3.2]. This property is established by combining known results from the single-particle localisation theory established via MSA [8] or FMM [1].  $\square$

**Theorem 7.** Let  $p > 0$  be fixed. There exists  $L_1^* < \infty$  such that for any  $L_0 \geq L_1^*$  and any  $k \geq 0$ , the estimate  $(\mathbf{DS}.k)_{m_k, p, L_0, E^*}$  holds true for any pair of  $L_k$ -distant non-diagonal cubes of size  $L_k$ .

*Proof.* Let  $I = [E^0, E^*]$ . We already prove  $(\mathbf{DS}.0)$  for some  $E^* > E^0$  and some  $m_0 > 0$ . Let  $k \geq 1$ . Let  $\mathbf{C}_{L_k}(\mathbf{x})$  and  $\mathbf{C}_{L_k}(\mathbf{y})$  be two non-diagonal  $L_k$ -distant cubes. We consider the events

$$\begin{aligned} B_k &= \{\exists E \in I : \mathbf{C}_{L_k}(\mathbf{x}), \mathbf{C}_{L_k}(\mathbf{y}) \text{ are both } (E, m_k)\text{-S}\}, \\ R &= \{\exists E \in I : \text{neither } \mathbf{C}_{L_k}(\mathbf{x}) \text{ nor } \mathbf{C}_{L_k}(\mathbf{y}) \text{ is } E\text{-CNR}\}, \\ T_{\mathbf{x}} &= \{\mathbf{C}_{L_k}(\mathbf{x}) \text{ is } 2m_k\text{-T}\}, \\ T_{\mathbf{y}} &= \{\mathbf{C}_{L_k}(\mathbf{y}) \text{ is } 2m_k\text{-T}\} \end{aligned}$$

Let  $\omega \in B_k \setminus R$ , then  $\forall E \in I$ ,  $\mathbf{C}_{L_k}(\mathbf{x})$ , or  $\mathbf{C}_{L_k}(\mathbf{y})$  is  $E$ -CNR. If  $\mathbf{C}_{L_k}(\mathbf{y})$  is  $E$ -CNR, then it must be  $2m_k$ -T: otherwise, it would have been  $(E, m_k)$ -NS by Lemma 5 applied for

$m = m_k$ . Similarly, if  $\mathbf{C}_{L_k}(\mathbf{x})$  is  $E$ -CNR, then it must be  $2m_k$ -T. This implies that

$$\mathbf{B} \subset \mathbf{R} \cup \mathbf{T}_{\mathbf{x}} \cup \mathbf{T}_{\mathbf{y}}. \quad (3.6)$$

We estimate  $\mathbb{P}\{\mathbf{R}\}$  using property  $(\mathbf{W2})_q$  which holds true by Lemma 1:

$$\mathbb{P}\{\mathbf{R}\} \leq L_k^{-q}.$$

We estimate  $\mathbb{P}\{\mathbf{T}_{\mathbf{x}}\}$ , and similarly  $\mathbb{P}\{\mathbf{T}_{\mathbf{y}}\}$ , by using Theorem 2:

$$\mathbb{P}\{\mathbf{T}_{\mathbf{x}}\} \leq \frac{(2L_k + 1)^{4d}}{2} L_{k-1}^{-2\tilde{p}}.$$

Hence,

$$\begin{aligned} \mathbb{P}\{\mathbf{B}_k\} &\leq \mathbb{P}\{\mathbf{R}\} + \mathbb{P}\{\mathbf{T}_{\mathbf{x}}\} + \mathbb{P}\{\mathbf{T}_{\mathbf{y}}\} \\ &\leq L_k^{-q} + 2 \frac{(2L_k + 1)^{4d}}{2} L_{k-1}^{-2\tilde{p}} \\ &\leq L_k^{-4p} + C(d) L_k^{4d - \frac{2\tilde{p}}{\alpha}} \quad (\text{since } q > 4p) \\ &\leq \frac{1}{2} L_k^{-2p} + \frac{1}{2} L_k^{-2p} \end{aligned}$$

since, by assumption,  $\tilde{p} > \frac{9}{4}p + \frac{15}{2}d$ .  $\square$

We end this section with a lemma on non-diagonal cubes which will be useful in the next sections.

**Lemma 6.** *Let  $\mathbf{C}_{L_{k+1}}(\mathbf{u})$  be a two-particle cube of size  $L_{k+1}$ . For  $E \in \mathbb{R}$ , we denote by*

- $M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{u}), E)$  the maximal number of pairwise  $L_k$ -distant, non-diagonal  $(E, m_k)$ -singular cubes  $\mathbf{C}_{L_k}(\mathbf{u}^{(j)}) \subset \mathbf{C}_{L_{k+1}}(\mathbf{u})$ .

Then,

$$\mathbb{P}\{\exists E \in I : M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{u}), E) \geq 2\} \leq \frac{(2L_{k+1} + 1)^{4d}}{2} \left( L_k^{-q} + C(d) L_k^{4d - \frac{2\tilde{p}}{\alpha}} \right). \quad (3.7)$$

*Proof.* The total number of possible pairs of centres  $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$  is bounded by  $\frac{1}{2}(2L_{k+1} + 1)^{4d}$ , while for a given pair of centres one can apply the probabilistic bound, i.e.,  $\mathbb{P}\{\mathbf{B}_k\}$  for a pair of  $L_k$ -distant non-diagonal cubes of sidelength  $2L_k$ .  $\square$

#### 4. CASE (II). DIAGONAL PAIRS OF SINGULAR CUBES

We assume here that for a pair of  $L_k$ -distant two-particle cubes  $\mathbf{C}_{L_k}(\mathbf{x})$  and  $\mathbf{C}_{L_k}(\mathbf{y})$  we have that  $\mathbb{P}(\mathbf{B}_k) \leq L_k^{-2p}$ .

Before we proceed further, let us state a geometric assertion borrowed from [4]. Given a two-particle cube  $\mathbf{C}_L(\mathbf{u}) = C_L(u_1) \times C_L(u_2) \subset \mathbb{Z}^d \times \mathbb{Z}^d$  we denote

$$\Pi \mathbf{C}_L(\mathbf{u}) = C_L(u_1) \cup C_L(u_2) \subset \mathbb{Z}^d \quad (4.1)$$

the union of the projections of  $\mathbf{C}_L(\mathbf{u})$  on the two factors of the product  $\mathbb{Z}^d \times \mathbb{Z}^d$ .

**Lemma 7** ([4, Lemma 4.1]). *Let  $L > r_0$ . If  $\mathbf{C}_L(\mathbf{u})$  and  $\mathbf{C}_L(\mathbf{v})$  are two diagonal  $L$ -distant two-particle cubes, then*

$$\Pi \mathbf{C}_L(\mathbf{u}) \cap \Pi \mathbf{C}_L(\mathbf{v}) = \emptyset. \quad (4.2)$$

*Proof.* See [4, Lemma 4.1].  $\square$

Lemma 7 is used in the proof of Lemma 8 which, in turn, is important in establishing the inductive step for a pair of distant diagonal cubes.

**Lemma 8.** *Assume that (DS.k) holds true for all pairs of  $L_k$ -distant diagonal cubes. Consider a two-particle cube  $\mathbf{C}_{L_{k+1}}(\mathbf{u})$ . For  $E \in \mathbb{R}$  we denote by*

- $M_D(\mathbf{C}_{L_{k+1}}(\mathbf{u}), E)$  *the maximal number of  $(E, m_k)$ -singular, pairwise  $L_k$ -distant diagonal cubes  $\mathbf{C}_{L_k}(\mathbf{u}^j) \subset \mathbf{C}_{L_{k+1}}(\mathbf{u})$ .*

*Then for all  $n \geq 1$ ,*

$$\mathbb{P} \{ \exists E \in I : M_D(\mathbf{C}_{L_{k+1}}(\mathbf{u}), E) \geq 2n \} \leq C(n, d) L_k^{4nd\alpha} L_k^{-2np}. \quad (4.3)$$

*Proof.* Suppose that there exist diagonal cubes  $\mathbf{C}_{L_k}(\mathbf{u}^j) \subset \mathbf{C}_{L_{k+1}}(\mathbf{x})$ ,  $1 \leq j \leq 2n$ , such that any two of them are  $L_k$ -distant. By Lemma 7, for any pair  $\mathbf{C}_{L_k}(\mathbf{u}^{2i-1})$ ,  $\mathbf{C}_{L_k}(\mathbf{u}^{2i})$ , the respective random operators  $\mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u}^{2i-1})}$  and  $\mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u}^{2i})}$  are independent and so are their spectra and Green functions. Moreover the pairs of operators

$$\left( \mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u}^{2i-1})}(\omega), \mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u}^{2i})}(\omega) \right), \quad i = 1, \dots, n,$$

form an independent family. The operator  $\mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u}^i)}$ , with  $i = 1, \dots, 2n$  is indeed measurable relative to the sigma-algebra  $\mathcal{B}_i$  generated by the random variables

$$\{V(x, \omega) : x \in \Pi \mathbf{C}_{L_k}(\mathbf{u}^i)\}.$$

Now by Lemma 7, the sets  $\Pi \mathbf{C}_{L_k}(\mathbf{u}^i)$ ,  $i = 1, \dots, 2n$ , are pairwise disjoint, so that all sigma-algebras  $\mathcal{B}_i$ ,  $i = 1, \dots, 2n$  are independent. Thus, any collection of events  $A_1, \dots, A_n$  relative to the corresponding pairs

$$\left( \mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u}^{2i-1})}(\omega), \mathbf{H}_{\mathbf{C}_{L_k}(\mathbf{u}^{2i})}(\omega) \right), \quad i = 1, \dots, n,$$

also form an independent family. For  $i = 1, \dots, n$ , set:

$$A_i = \{ \exists E \in I : \mathbf{C}_{L_k}(\mathbf{u}^{2i-1}) \text{ and } \mathbf{C}_{L_k}(\mathbf{u}^{2i+2}) \text{ are } (E, m)\text{-S} \}.$$

Then by virtue of the inductive assumption,

$$\mathbb{P}(A_i) \leq L_k^{-2p}, \quad 1 \leq i \leq n, \quad (4.4)$$

and owing to independence of events  $A_1, \dots, A_n$ , we obtain

$$\mathbb{P} \left\{ \bigcap_{i=1}^n A_i \right\} = \prod_{j=1}^n \mathbb{P}(A_j) \leq \left( L_k^{-2p} \right)^n. \quad (4.5)$$

To complete the proof note that the total number of different families of  $2n$  cubes  $\mathbf{C}_{L_k} \subset \mathbf{C}_{L_{k+1}}(\mathbf{x})$  with required properties is bounded from above by

$$\frac{1}{(2n)!} |\mathbf{C}_{L_{k+1}}(\mathbf{u})|^{2n} \leq C(n, d) L_k^{4dn\alpha}. \quad \square$$

**Lemma 9.** *Let  $J \geq 1$  be an odd integer and  $E \in \mathbb{R}$ . Let  $\mathbf{C}_{L_{k+1}}(\mathbf{u})$  be a cube such that:*

- (i)  $\mathbf{C}_{L_{k+1}}(\mathbf{u})$  *is  $E$ -CNR,*
- (ii)  $M_{ND}(\mathbf{C}_{L_{k+1}}(\mathbf{u}), E) + M_D(\mathbf{C}_{L_{k+1}}(\mathbf{u}), E) \leq J$ .

*Then, for  $L_0$  large enough  $\mathbf{C}_{L_{k+1}}(\mathbf{u})$  is  $(E, m_{k+1})$ -NS for some  $m_{k+1}$  such that*

$$m_{k+1} \geq m_k \left( 1 - \frac{5J+6}{(2L_k)^{1/2}} \right).$$

*Proof.* Simple reformulation of [5, Lemma 4.2]. See also [4, Lemma 4.5].  $\square$

**Theorem 8.** *There exists  $0 < L_2^* < +\infty$  such that if  $L_0 \geq L_2^*$ , then if for  $k \geq 0$ , property **(DS.k)** holds true for all pairs of  $L_k$ -distant diagonal cubes  $\mathbf{C}_{L_k}(\mathbf{x})$ ,  $\mathbf{C}_{L_k}(\mathbf{y})$ , then **(DS.k + 1)** holds true for all pairs of  $L_{k+1}$ -distant diagonal cubes  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$ ,  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$*

*Proof.* Consider a pair of  $L_{k+1}$ -distant two-particle diagonal cubes  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  and  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$ . Let us set

$$\begin{aligned} B_{k+1} &= \{\exists E \in I : \mathbf{C}_{L_{k+1}}(\mathbf{x}), \text{ and } \mathbf{C}_{L_{k+1}}(\mathbf{y}) \text{ are } (E, m_{k+1})\text{-S}\} \\ \Sigma &= \{\exists E \in I : \text{neither } \mathbf{C}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{C}_{L_{k+1}}(\mathbf{y}) \text{ is } E\text{-CNR}\} \\ S_{\mathbf{x}} &= \{\exists E \in I : M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}), E) + M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}), E) \geq J + 1\} \\ S_{\mathbf{y}} &= \{\exists E \in I : M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}), E) + M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}), E) \geq J + 1\}. \end{aligned}$$

Let  $\omega \in B_{k+1}$ . Suppose that  $\omega \notin \Sigma \cup S_{\mathbf{x}}$  so that  $\forall E \in I$  either  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  or  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$  is  $(E, J)$ -CNR and  $M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}), E) + M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}), E) \leq J$ . Thus the cube  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  cannot be  $(E, J)$ -CNR: indeed, by Lemma 9, it would be  $(E, m_{k+1})$ -NS. So, it is the cube  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$  which is  $(E, J)$ -CNR and  $(E, m_{k+1})$ -S. This implies again by Lemma 9 that

$$M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}); E) + M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}); E) \geq J + 1.$$

Therefore  $\omega \in S_{\mathbf{y}}$ . This shows that

$$B_{k+1} \subset \Sigma \cup S_{\mathbf{x}} \cup S_{\mathbf{y}}.$$

Therefore,

$$\begin{aligned} \mathbb{P}\{B_{k+1}\} &\leq \mathbb{P}\{\Sigma\} + \mathbb{P}\{S_{\mathbf{x}}\} + \mathbb{P}\{S_{\mathbf{y}}\} \\ &\leq L_{k+1}^{-q} + 2\mathbb{P}\{S_{\mathbf{x}}\}. \end{aligned}$$

with  $q > 4p$  large enough.

It remains to estimate  $\mathbb{P}\{S_{\mathbf{x}}\}$ . Set  $J = 2n + 1$ , then

$$M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}); E) + M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}); E) \geq 2n + 2$$

implies that either  $M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}); E) \geq 2$  or  $M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}); E) \geq 2n$ . Then by Lemma 6 and Lemma 8, we have

$$\begin{aligned} \mathbb{P}\{S_{\mathbf{x}}\} &\leq \mathbb{P}\{\exists E \in I : M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}); E) \geq 2\} + \mathbb{P}\{\exists E \in I : M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{x}); E) \geq 2n\} \\ &\leq \frac{(2L_{k+1} + 1)^{4d}}{2} (L_k^{-q} + C(d)L_k^{4d - \frac{2\tilde{p}}{\alpha}}) + C(n, d)L_k^{4dn\alpha - 2np} \\ &\leq C(d) \left( L_{k+1}^{4d - \frac{4p}{\alpha}} + L_{k+1}^{4d + \frac{4d}{\alpha} - \frac{2\tilde{p}}{\alpha^2}} \right) + C(n, d)L_{k+1}^{4nd - \frac{2np}{\alpha}} \\ &\leq C(d) \left( L_{k+1}^{4d - \frac{8p}{3}} + L_{k+1}^{-\frac{8\tilde{p}}{9} + \frac{8d}{9} + 4d} + L_{k+1}^{-\frac{4p}{\alpha} + 8d} \right) \quad (\text{by taking } n = 2 \text{ and } \alpha = 3/2) \\ &\leq L_{k+1}^{-2p}, \end{aligned}$$

where we used that by our assumptions  $q > 4p$ ,  $p > 12d$  and  $\tilde{p} > \frac{9}{4}p + \frac{15}{2}d$ .  $\square$

## 5. CASE (III). MIXED PAIRS OF SINGULAR CUBES

Now we derive property **(DS.k + 1)** in the case (III), for mixed pairs of two-particle cubes (where one cube is diagonal and the other non-diagonal). Here we use several properties which have been established earlier in this paper for all scales lengths. Namely **(W1)**, **(W2)**, **(NDRoNS)** and the inductive assumption.

**Theorem 9.** *There exists  $0 < L_3^* < +\infty$  such that if  $L_0 \geq L_3^*$  and if for  $k \geq 0$ , property (DS.k) holds true*

- (i) *for any pair of  $L_k$ -distant non-diagonal cubes  $\mathbf{C}_{L_k}(\mathbf{x})$ ,  $\mathbf{C}_{L_k}(\mathbf{y})$ ,*
- (ii) *for any pair of  $L_k$ -distant diagonal cubes  $\mathbf{C}_{L_k}(\mathbf{x})$ ,  $\mathbf{C}_{L_k}(\mathbf{y})$ .*

*Then (DS.k+1) holds true for all mixed pairs of  $L_{k+1}$ -distant cubes  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$ ,  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$ .*

*Proof.* Consider a pair of two particles  $L_{k+1}$ -distant cubes  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  and  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$ , where  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  is ND while  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$  is D. Let us set as in the previous sections

$$\begin{aligned} B_{k+1} &= \{\exists E \in I : \mathbf{C}_{L_{k+1}}(\mathbf{x}), \mathbf{C}_{L_{k+1}}(\mathbf{y}), \text{ are } (E, m_{k+1})\text{-S}\} \\ \Sigma &= \{\exists E \in I : \text{neither } \mathbf{C}_{L_{k+1}}(\mathbf{x}) \text{ nor } \mathbf{C}_{L_{k+1}}(\mathbf{y}) \text{ is } E\text{-CNR}\} \\ T_{\mathbf{x}} &= \{\mathbf{C}_{L_{k+1}}(\mathbf{x}) \text{ is } 2m_{k+1}\text{-T}\} \\ S_{\mathbf{y}} &= \{\exists E \in I : M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}); E) + M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}); E) \geq J + 1\}. \end{aligned}$$

Let  $\omega \in B_{k+1} \setminus (\Sigma \cup T_{\mathbf{x}})$ , then  $\forall E \in I$  either  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  or  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$  is  $(E, J)$ -CNR and  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  is  $2m_{k+1}$ -NT. By Lemma 5,  $\mathbf{C}_{L_{k+1}}(\mathbf{x})$  cannot be  $(E, J)$ -CNR. Indeed it would have been  $(E, m_{k+1})$ -NS. So it is the cube  $\mathbf{C}_{L_{k+1}}(\mathbf{y})$  which is  $(E, J)$ -CNR. By Lemma 9, this implies  $M_{\text{ND}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}); E) + M_{\text{D}}(\mathbf{C}_{L_{k+1}}(\mathbf{y}); E) \geq J + 1$  for some  $E \in I$ . Therefore  $\omega \in S_{\mathbf{y}}$ . This shows that

$$B_{k+1} \subset \Sigma \cup T_{\mathbf{x}} \cup S_{\mathbf{y}}.$$

Then, using estimates of  $\mathbb{P}\{\Sigma\}$ ,  $\mathbb{P}\{T_{\mathbf{x}}\}$  and  $\mathbb{P}\{S_{\mathbf{y}}\}$  already established in previous sections,

$$\begin{aligned} \mathbb{P}\{B_{k+1}\} &\leq \mathbb{P}\{\Sigma\} + \mathbb{P}\{T_{\mathbf{x}}\} + \mathbb{P}\{S_{\mathbf{y}}\} \\ &\leq L_{k+1}^{-q} + \frac{1}{2}(2L_{k+1} + 1)^{4d} L_{k+1}^{-\frac{2\bar{p}}{\alpha}} + \frac{1}{4} L_{k+1}^{-2p} \\ &\leq L_{k+1}^{-4p} + C(d) L_{k+1}^{4d - \frac{2\bar{p}}{\alpha}} + \frac{1}{4} L_{k+1}^{-2p} \\ &\leq \frac{1}{4} L_{k+1}^{-2p} + \frac{1}{2} L_{k+1}^{-2p} + \frac{1}{4} L_{k+1}^{-2p} \\ &\leq L_{k+1}^{-2p}. \quad \square \end{aligned}$$

This completes the proof of Theorem 6. Therefore, Theorem 3 is also proven, since we have already proven Theorem 5 giving the base of induction in  $k$ . By virtue of Theorem 4, this completes also the proof of our main result on two-particle localization at low energies, Theorem 1.

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